Méthode de CROUT (Décomposition LU)

Soit le système $Ax = y$
Le principe est de décomposer la matrice A en matrices triangulaires

$A = L\,U$

Triang inf — Triang sup diago 1

$$Ax = y$$

$$LU\,x = y$$

on pose $Z = Ux$

$$\begin{cases} LZ = x \\ Ux = Z \end{cases}$$

Exp: Résoudre le système suivant en utilisant la méthode de CROUT

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 9 & 27 \\ 2 & 4 & 8 \end{pmatrix} x = \begin{pmatrix} 14 \\ 120 \\ 50 \end{pmatrix}$$

$\Rightarrow$ Sol

Détermination des matrices LU : $A = LU$

$$\begin{pmatrix} 1 & u_{01} & u_{02} \\ 0 & 1 & u_{12} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} l_{00} & 0 & 0 \\ l_{10} & l_{11} & 0 \\ l_{20} & l_{21} & l_{22} \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 3 & 9 & 27 \\ 2 & 4 & 8 \end{pmatrix}$$

$l_{00} = 1$

$l_{00} u_{01} = 1 \Rightarrow u_{01} = 1$

$l_{00} u_{02} = 1 \Rightarrow u_{02} = 1$

$l_{10} = 3$

$l_{10} u_{01} + l_{11} = 9 \Rightarrow l_{11} = 6$

$l_{10} \times u_{02} + l_{11} \times u_{12} = 27$

$3 \times 1 + 6 \times u_{12} = 27$

$u_{12} = \frac{24}{6} \Rightarrow u_{12} = 4$

$l_{20} = 2$

$l_{20} u_{01} + l_{21} = 4$

$2 + l_{21} = 4 \Rightarrow l_{21} = 2$

$l_{20} u_{02} + l_{21} u_{12} + l_{22} = 8$

$2 + 8 + l_{22} = 8$

$\Rightarrow l_{22} = -2$

donc $= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 2 & 2 & -2 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$

On pose $z = Ux$

$Ax = y$

$L\underbrace{Ux}_{z} = y$

on doit resoudre :

$Lz = y \ldots (1)$

$Ux = z \ldots (2)$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 2 & 2 & -2 \end{pmatrix}}_{L} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 14 \\ 120 \\ 50 \end{pmatrix}}_{y}$$

Par sub avant

$z_0 = 14$, $z_1 = 13$, $z_2 = 2 \Rightarrow z = \begin{pmatrix} 14 \\ 13 \\ 2 \end{pmatrix}$

$Ux = z$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 2 \end{pmatrix}$$

$$\begin{array}{l} L_0 \\ L_1 \\ L_2 - 3L_1 \end{array} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

$$\begin{array}{l} L_0 \\ L_1 / -2 \\ L_2 / -5 \end{array} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$x_2 = 1$

$x_1 = -2$

$x_0 + 2x_1 + 4x_2 = 3$

$x_0 + -4 + 4 = 3$

$x_0 = 3$

$x_0 = 3$

$$x = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

on a :

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 8 \\ 3 & 6 & 7 \end{pmatrix} = \begin{pmatrix} l_{00} & 0 & 0 \\ l_{10} & l_{11} & 0 \\ l_{20} & l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$l_{00} = 1$ $\quad$ $l_{10} = 2$ $\quad$ $2l_{10} + l_{11} = 2 \Rightarrow l_{1}$

$l_{20} = 3$ $\quad$ $2l_{20} + l_{21} = 6 \Rightarrow l_{21} = 0$

$4l_{20} + l_{22} = 7 \Rightarrow l_{22} = -5$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 5 \end{pmatrix}$$

on pose

$U.x = z$ $\quad$ $Ax = y$

$L \underbrace{Ux}_{z} = y$

on doit résoudre $Lz = y$ (1)

$Ux = z$ (2)

(1) $Lz = y$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & -5 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 4 \end{pmatrix}$$

$z_0 = 3$

$z_1 = (10 - 6) / -2 = -2$

$z_3 = (4 - 9) / -5 = 1$

$$z = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$Ux = z$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$x_2 = 1$

$x_1 = -2$

$x_0 = 3 + 4 - 4 \Rightarrow x_0 = 3$

$$x = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$A^2 x = y$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & -5 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$A^2 x = y$

$A * A * x = y$

on pose : $T = Ax$

$\begin{cases} AT = y & — (1) \\ T = Ax & — (2) \end{cases}$

(1) $\underset{LU}{A}T = y \Rightarrow T = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$Ax = y$

(2) $\underset{LU}{A}x = T$

on a $A = LU$

$\underbrace{LUx}_{z_n} = T$

on pose $z_n = Ux$

on doit résoudre

$Lz_n = T$

$Ux = z_n$

$\Rightarrow x = \begin{pmatrix} -57/5 \\ 4 \\ 8/5 \end{pmatrix}$